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Closed conformal Killing-Yano tensor and geodesic integrability

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Abstract

Assuming the existence of a single rank-2 closed conformal Killing-Yano tensor with a certain symmetry we show that there exist mutually commuting rank-2 Killing tensors and Killing vectors. We also discuss the condition of separation of variables for the geodesic Hamilton-Jacobi equations.

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1 Introduction

Recently, it has been shown that geodesic motion in the Kerr-NUT de Sitter spacetime is integrable for all dimensions [1, 2, 3, 4, 5, 6]. Indeed, the constants of motion that are in involution can be explicitly constructed from a rank-2 closed conformal Killing-Yano (CKY) tensor. In this paper we consider the problem of integrability of the geodesic equation in a more general situation. We assume the existence of a single rank-2 closed CKY tensor with a certain symmetry for D -dimensional spacetime M with a metric g . It turns out that such a spacetime admits mutually commuting k rank-2 Killing tensors and k Killing vectors. Here we put $D = 2k$ for even D , and $D = 2k - 1$ for odd D . Although the existence of the commuting Killing tensors was shown in [5, 6], we reproduce it more directly. We also discuss the condition of separation of variables for the geodesic Hamilton-Jacobi equations using the result given by Benenti-Francaviglia [7] and Kalnins-Miller [8] (see also [9]).

2 Assumptions and main results

A two-form

$$h = \frac{1}{2}h_{ab}dx^a \wedge dx^b, \quad h_{ab} = -h_{ba} \quad (2.1)$$

is called a conformal Killing-Yano (CKY) tensor if it satisfies

$$\nabla_a h_{bc} + \nabla_b h_{ac} = 2\xi_c g_{ab} - \xi_a g_{bc} - \xi_b g_{ac}. \quad (2.2)$$

The vector field ξ_a is called the associated vector of h_{ab} , which is given by

$$\xi_a = \frac{1}{D-1} \nabla^b h_{ba}. \quad (2.3)$$

In the following we assume

$$(a1) \ dh = 0, \quad (a2) \ \mathcal{L}_\xi g = 0, \quad (a3) \ \mathcal{L}_\xi h = 0. \quad (2.4)$$

The assumption (a1) means that $(D-2)$ -form $f = *h$ is a Killing-Yano (KY) tensor,

$$\nabla_{(a_1} f_{a_2) a_3 \dots a_{D-1}} = 0. \quad (2.5)$$

Note that the equation (2.2) together with (a1) is equivalent to

$$\nabla_a h_{bc} = \xi_c g_{ab} - \xi_b g_{ac}. \quad (2.6)$$

It was shown in [10] that the associated vector ξ satisfies

$$\nabla_a \xi_b + \nabla_b \xi_a = \frac{1}{D-2} (R_a^c h_{bc} + R_b^c h_{ac}), \quad (2.7)$$

where R_{ab} is a Ricci tensor. If M is Einstein, i.e. $R_{ab} = \Lambda g_{ab}$, then

$$\nabla_a \xi_b + \nabla_b \xi_a = 0. \quad (2.8)$$

Thus, any Einstein space satisfies the assumption (a2) [10]. According to [5], we define $2j$ -forms $h^{(j)}$ ($j = 0, \dots, k-1$):

$$h^{(j)} = \underbrace{h \wedge h \wedge \dots \wedge h}_j = \frac{1}{(2j)!} h_{a_1 \dots a_{2j}} dx^{a_1} \wedge \dots \wedge dx^{a_{2j}}, \quad (2.9)$$

where the components are written as

$$h_{a_1 \dots a_{2j}}^{(j)} = \frac{(2j)!}{2^j} h_{[a_1 a_2} h_{a_3 a_4} \dots h_{a_{2j-1} a_{2j}]}. \quad (2.10)$$

Since the wedge product of two CKY tensors is again a CKY tensor, $h^{(j)}$ are closed CKY tensors, and so $f^{(j)} = *h^{(j)}$ KY tensors. Explicitly, we have

$$f^{(j)} = *h^{(j)} = \frac{1}{(D-2j)!} f_{a_1 \dots a_{D-2j}}^{(j)} dx^{a_1} \wedge \dots \wedge dx^{a_{D-2j}}, \quad (2.11)$$

where

$$f_{a_1 \dots a_{D-2j}}^{(j)} = \frac{1}{(2j)!} \varepsilon^{b_1 \dots b_{2j}} h_{b_1 \dots b_{2j}}^{(j)} \delta_{a_1 \dots a_{D-2j}}^{b_1 \dots b_{2j}}. \quad (2.12)$$

Given these KY tensors, we can construct the rank-2 Killing tensors $K^{(j)}$ obeying the equation $\nabla_{(a} K_{bc)}^{(j)} = 0$:

$$K_{ab}^{(j)} = \frac{1}{(D-2j-1)!(j!)^2} f_{ac_1 \dots c_{D-2j-1}}^{(j)} f_b^{(j) c_1 \dots c_{D-2j-1}}. \quad (2.13)$$

From (a2) we have $\mathcal{L}_\xi *h^{(j)} = *\mathcal{L}_\xi h^{(j)}$ and hence the assumption (a3) yields

$$\mathcal{L}_\xi h^{(j)} = 0, \quad \mathcal{L}_\xi f^{(j)} = 0, \quad \mathcal{L}_\xi K^{(j)} = 0. \quad (2.14)$$

We also immediately obtain from (2.6)

$$\nabla_\xi h^{(j)} = 0, \quad \nabla_\xi f^{(j)} = 0, \quad \nabla_\xi K^{(j)} = 0. \quad (2.15)$$

Let us define the vector fields $\eta^{(j)}$ by [11, 12]

$$\eta_a^{(j)} = K_a^{(j) b} \xi_b. \quad (2.16)$$

Then we have

$$\nabla_{(a}\eta_{b)}^{(j)} = \frac{1}{2}\mathcal{L}_\xi K_{ab}^{(j)} - \nabla_\xi K_{ab}^{(j)}, \quad (2.17)$$

which vanishes by (2.14) and (2.15), i.e. $\eta^{(j)}$ are Killing vectors.

Theorem 1 was proved in [5, 6].

Theorem 1 Under (a1) Killing tensors $K^{(i)}$ are mutually commuting,

$$[K^{(i)}, K^{(j)}]_S = 0.$$

The bracket $[\ , \]_S$ represents a symmetric Schouten product. The equation can be written as

$$K_{d(a}^{(i)} \nabla^d K_{bc)}^{(j)} - K_{d(a}^{(j)} \nabla^d K_{bc)}^{(i)} = 0. \quad (2.18)$$

Adding the assumptions (a2) and (a3) we prove

Theorem 2

$$\mathcal{L}_{\eta^{(i)}} h = 0. \quad (2.19)$$

Corollary Killing vectors $\eta^{(i)}$ and Killing tensors $K^{(j)}$ are mutually commuting,

$$[\eta^{(i)}, K^{(j)}]_S = 0, \quad [\eta^{(i)}, \eta^{(j)}] = 0.$$

3 Proof of theorems 1,2

Let $H, Q := -H^2, K^{(j)}$ be matrices with elements

$$H^a_b = h^a_b, \quad Q^a_b = -h^a_c h^c_b, \quad (K^{(j)})^a_b = K^{(j)a}_b. \quad (3.1)$$

The generating function of $K^{(j)}$ can be read off from [5]:

$$K_{ab}(\beta) = \sum_{j=0}^{k-1} K_{ab}^{(j)} \beta^j = \det^{1/2}(I + \beta Q) \left[(I + \beta Q)^{-1} \right]_{ab}. \quad (3.2)$$

Here $k = [(D+1)/2]$. Note that

$$\begin{aligned} & 2 \det^{1/2}(I + \beta Q) \left[(I + \beta Q)^{-1} \right]_b^a \\ &= \det(I + \sqrt{\beta} H) \left[(I + \sqrt{\beta} H)^{-1} \right]_b^a + \det(I - \sqrt{\beta} H) \left[(I - \sqrt{\beta} H)^{-1} \right]_b^a. \end{aligned} \quad (3.3)$$

Since $\det(I \pm \sqrt{\beta} H) [(I \pm \sqrt{\beta} H)^{-1}]_b^a$ is a cofactor of the matrix $I \pm \sqrt{\beta} H$, (3.2) is indeed a polynomial of β of degree $[(D-1)/2]$.

For simplicity, let us define a matrix $S(\beta)$ by

$$S(\beta) := (I + \beta Q)^{-1}. \quad (3.4)$$

Using (2.6), we have

$$\nabla_a \det^{1/2}(I + \beta Q) = -2\beta \xi_d \left[HS(\beta) \right]_a^d \det^{1/2}(I + \beta Q), \quad (3.5)$$

$$\begin{aligned} \nabla_a S_{bc}(\beta) &= \beta S_{ba}(\beta) \xi^d \left[HS(\beta) \right]_{dc} - \beta S_{bd}(\beta) \xi^d \left[HS(\beta) \right]_{ac} \\ &\quad + \beta \left[HS(\beta) \right]_{ba} \xi^d S_{dc}(\beta) - \beta \left[HS(\beta) \right]_{bd} \xi^d S_{ac}(\beta). \end{aligned} \quad (3.6)$$

Combining these relations, we have

$$\nabla_a K_{bc}(\beta) = \det^{1/2}(I + \beta Q) \xi^d X_{abc;d}(\beta), \quad (3.7)$$

where

$$\begin{aligned} X_{abc;d}(\beta) &= 2\beta \left[HS(\beta) \right]_{ad} S_{bc}(\beta) - \beta \left[HS(\beta) \right]_{bd} S_{ca}(\beta) - \beta \left[HS(\beta) \right]_{cd} S_{ab}(\beta) \\ &\quad + \beta S_{bd}(\beta) \left[HS(\beta) \right]_{ca} + \beta S_{cd}(\beta) \left[HS(\beta) \right]_{ba}. \end{aligned} \quad (3.8)$$

Then with help of (3.7), it is easy to check that the following relations hold:

$$\nabla_{(a} K_{bc)}(\beta) = 0. \quad (3.9)$$

Therefore we have

$$\nabla_{(a} K_{bc)}^{(j)} = 0. \quad (3.10)$$

Proof of Theorem 1. In terms of generating function, Theorem 1 (2.18) can be written as follows

$$K_{e(a}(\beta_1) \nabla^e K_{bc)}(\beta_2) - K_{e(a}(\beta_2) \nabla^e K_{bc)}(\beta_1) = 0. \quad (3.11)$$

Let

$$F_{abc}(\beta_1, \beta_2) := \frac{K_{ea}(\beta_1) \nabla^e K_{bc}(\beta_2)}{\det^{1/2}(I + \beta_1 Q) \det^{1/2}(I + \beta_2 Q)}. \quad (3.12)$$

(3.11) is equivalent to

$$F_{(abc)}(\beta_1, \beta_2) - F_{(abc)}(\beta_2, \beta_1) = 0. \quad (3.13)$$

Using the explicit form of $\nabla^e K_{bc}(\beta_2)$, we have

$$\begin{aligned}
F_{abc}(\beta_1, \beta_2) &= \beta_2 \xi^d S_{ea}(\beta_1) \\
&\times \left(2[HS(\beta_2)]_{ed} S_{bc}(\beta_2) - [HS(\beta_2)]_{bd} S_c^e(\beta_2) - [HS(\beta_2)]_{cd} S_b^e(\beta_2) \right. \\
&\quad \left. + S_{bd}(\beta_2) [HS(\beta_2)]_c^e + S_{cd}(\beta_2) [HS(\beta_2)]_b^e \right) \\
&= \beta_2 \xi^d \left(2[HS(\beta_1)S(\beta_2)]_{ad} S_{bc}(\beta_2) \right. \\
&\quad - [HS(\beta_2)]_{bd} [S(\beta_1)S(\beta_2)]_{ca} - [HS(\beta_2)]_{cd} [S(\beta_1)S(\beta_2)]_{ab} \\
&\quad \left. + S_{bd}(\beta_2) [HS(\beta_1)S(\beta_2)]_{ca} + S_{cd}(\beta_2) [HS(\beta_1)S(\beta_2)]_{ba} \right). \tag{3.14}
\end{aligned}$$

Then

$$F_{(abc)}(\beta_1, \beta_2) = 2\beta_2 \xi^d \left(S_{(bc}(\beta_2) [HS(\beta_1)S(\beta_2)]_{a)d} - [S(\beta_1)S(\beta_2)]_{(bc} [HS(\beta_2)]_{a)d} \right). \tag{3.15}$$

Note that

$$\beta_2 S(\beta_2) - \beta_1 S(\beta_1) = (\beta_2 - \beta_1) S(\beta_1) S(\beta_2). \tag{3.16}$$

Then

$$\begin{aligned}
&F_{(abc)}(\beta_1, \beta_2) - F_{(abc)}(\beta_2, \beta_1) \\
&= 2(\beta_2 - \beta_1) \xi^d \left([S(\beta_1)S(\beta_2)]_{(bc} [HS(\beta_1)S(\beta_2)]_{a)d} - [S(\beta_1)S(\beta_2)]_{(bc} [HS(\beta_1)S(\beta_2)]_{a)d} \right) \\
&= 0.
\end{aligned}$$

This completes the proof of Theorem 1. \square

Let $\eta_a(\beta)$ be the generating function of $\eta_a^{(j)}$:

$$\eta_a(\beta) = \sum_{j=0}^{k-1} \eta_a^{(j)} \beta^j = K_{ab}(\beta) \xi^b. \tag{3.17}$$

Proof of Theorem 2. In terms of the generating function (3.17), the theorem 2 is equivalent to

$$\mathcal{L}_{\eta(\beta)} h_{ab} = 0. \tag{3.18}$$

The left-handed side is

$$\mathcal{L}_{\eta(\beta)} h_{ab} = \eta^c(\beta) \nabla_c h_{ab} + h_{cb} \nabla_a \eta^c(\beta) + h_{ac} \nabla_b \eta^c(\beta). \tag{3.19}$$

Using (2.6), the first term in the right-handed side of (3.19) becomes

$$\eta^c(\beta) \nabla_c h_{ab} = \xi_b \eta_a(\beta) - \xi_a \eta_b(\beta). \tag{3.20}$$

Let us examine the second and third terms.

$$\begin{aligned}
U_{ab}(\beta) &:= h_{cb} \nabla_a \eta^c(\beta) + h_{ac} \nabla_b \eta^c(\beta) \\
&= h_{cb} \nabla_a (K^c_d(\beta) \xi^d) + h_{ac} \nabla_b (K^c_d(\beta) \xi^d) \\
&= [K(\beta)H]_{db} \nabla_a \xi^d + [K(\beta)H]_{ad} \nabla_b \xi^d + \xi^d (h_{cb} \nabla_a K^c_d(\beta) + h_{ac} \nabla_b K^c_d(\beta)).
\end{aligned} \tag{3.21}$$

Note that

$$[K(\beta)H]_{db} \nabla_a \xi^d + [K(\beta)H]_{ad} \nabla_b \xi^d = \mathcal{L}_\xi [K(\beta)H]_{ab} - \nabla_\xi [K(\beta)H]_{ab} = 0. \tag{3.22}$$

Here we have used (2.14) and (2.15).

Let

$$V_{ab}(\beta) := \frac{\xi^d h_{ac} \nabla_b K^c_d(\beta)}{\det^{1/2}(I + \beta Q)}. \tag{3.23}$$

Then

$$U_{ab}(\beta) = \det^{1/2}(I + \beta Q) (V_{ab}(\beta) - V_{ba}(\beta)) = 2 \det^{1/2}(I + \beta Q) V_{[ab]}(\beta). \tag{3.24}$$

Using (3.7), we have

$$V_{ab}(\beta) = \beta \xi^d \xi^f \left\{ [HS(\beta)]_{ad} [HS(\beta)]_{bf} - S_{df} [QS(\beta)]_{ab} + [QS(\beta)]_{ad} S_{bf}(\beta) \right\}, \tag{3.25}$$

$$2V_{[ab]}(\beta) = \beta \xi^d \xi^f \left\{ [QS(\beta)]_{ad} S_{bf}(\beta) - S_{ad}(\beta) [QS(\beta)]_{bf} \right\}. \tag{3.26}$$

Note that

$$\beta QS(\beta) = I - S(\beta). \tag{3.27}$$

Then

$$2V_{[ab]}(\beta) = \beta \xi^d \xi^f \left\{ g_{ad} S_{bf}(\beta) - S_{ad}(\beta) g_{bf} \right\} = \xi_a S_{bf}(\beta) \xi^f - \xi_b S_{ad}(\beta) \xi^d. \tag{3.28}$$

Therefore

$$U_{ab}(\beta) = \xi_a \eta_b(\beta) - \xi_b \eta_a(\beta). \tag{3.29}$$

Adding (3.20) and (3.29), we have

$$\mathcal{L}_{\eta(\beta)} h_{ab} = 0. \tag{3.30}$$

This completes the proof of Theorem 2. \square

The first relation of Corollary is equivalent to

$$\mathcal{L}_{\eta^{(i)}} K^{(j)} = 0, \tag{3.31}$$

which immediately follows from Theorem 2.

The second relation of Corollary is equivalent to

$$\mathcal{L}_{\eta^{(i)}}\eta^{(j)} = 0. \quad (3.32)$$

Note that

$$\mathcal{L}_\xi \xi = [\xi, \xi] = 0, \quad (3.33)$$

$$\begin{aligned} \mathcal{L}_\xi \eta^{(j)a} &= \mathcal{L}_\xi (K^{(j)a}{}_b \xi^b) \\ &= (\mathcal{L}_\xi K^{(j)a}{}_b) \xi^b + K^{(j)a}{}_b (\mathcal{L}_\xi \xi^b) \\ &= 0. \end{aligned} \quad (3.34)$$

Here we have used (2.14) and (3.33). Then

$$\mathcal{L}_{\eta^{(j)}}\xi = [\eta^{(j)}, \xi] = -\mathcal{L}_\xi \eta^{(j)} = 0. \quad (3.35)$$

Now, using this relation and (3.31), we easily see that

$$\begin{aligned} \mathcal{L}_{\eta^{(i)}}\eta^{(j)a} &= \mathcal{L}_{\eta^{(i)}}(K^{(j)a}{}_b \xi^b) \\ &= (\mathcal{L}_{\eta^{(i)}} K^{(j)a}{}_b) \xi^b + K^{(j)a}{}_b (\mathcal{L}_{\eta^{(i)}} \xi^b) \\ &= 0. \end{aligned} \quad (3.36)$$

This completes the proof of Corollary.

4 Separation of variables in the Hamilton-Jacobi equation

A geometric characterisation of the separation of variables in the geodesic Hamilton-Jacobi equation was given by Benenti-Francaviglia [7] and Kalnins-Miller [8]. Here, we use the following result in [8].

Theorem Suppose there exists a N -dimensional vector space \mathcal{A} of rank-2 Killing tensors on D -dimensional space (M, g) . Then the geodesic Hamilton-Jacobi equation has a separable coordinate system if and only if the following conditions hold¹:

- (1) $[A, B]_S = 0$ for each $A, B \in \mathcal{A}$.
- (2) There exist $(D - n)$ -independent simultaneous eigenvectors $X^{(a)}$ for every $A \in \mathcal{A}$.

¹We put $n_2 = 0$ for theorem 4 in [8]. This condition is satisfied in the case of a positive definite metric g .

- (3) There exist n -independent commuting Killing vectors $Y^{(\alpha)}$.
- (4) $[A, Y^{(\alpha)}]_S = 0$ for each $A \in \mathcal{A}$.
- (5) $N = (2D + n^2 - n)/2$.
- (6) $g(X^{(a)}, X^{(b)}) = 0$ if $1 \leq a < b \leq D - n$,
and $g(X^{(a)}, Y^{(\alpha)}) = 0$ for $1 \leq a \leq D - n$, $D - n + 1 \leq \alpha \leq D$.

We assume that the Killing tensors $K^{(j)}$ and $K^{(ij)} = \eta^{(i)} \otimes \eta^{(j)} + \eta^{(j)} \otimes \eta^{(i)}$ given in section 2 form a basis for \mathcal{A} . Note that in the odd dimensional case the last Killing Yano tensor $f^{(k-1)}$ is a Killing vector, and hence the corresponding Killing tensor $K^{(k-1)} \propto f^{(k-1)} f^{(k-1)}$ is reducible [5]. Then, it is easy to see that the conditions (1) \sim (6) hold. Indeed, the relation $K^{(i)} K^{(j)} = K^{(j)} K^{(i)}$ implies that there exist simultaneous eigenvectors $X^{(a)}$ for $K^{(i)}$ satisfying conditions (2) and (6). Other conditions are direct consequences of Theorem 1 and Corollary.

5 Example

Finally we describe the Kerr-NUT de Sitter metric as an example, which was fully studied in [13, 14, 1, 2, 3, 4, 5, 6]. The D -dimensional metric takes the form [13]:

(a) $D = 2n$

$$g = \sum_{\mu=1}^n \frac{dx_\mu^2}{Q_\mu} + \sum_{\mu=1}^n Q_\mu \left(\sum_{k=0}^{n-1} A_\mu^{(k)} d\psi_k \right)^2 \quad (5.1)$$

(b) $D = 2n + 1$

$$g = \sum_{\mu=1}^n \frac{dx_\mu^2}{Q_\mu} + \sum_{\mu=1}^n Q_\mu \left(\sum_{k=0}^{n-1} A_\mu^{(k)} d\psi_k \right)^2 + S \left(\sum_{k=0}^n A^{(k)} d\psi_k \right)^2 \quad (5.2)$$

The functions Q_μ are given by

$$Q_\mu = \frac{X_\mu}{U_\mu}, \quad U_\mu = \prod_{\substack{\nu=1 \\ (\nu \neq \mu)}}^n (x_\mu^2 - x_\nu^2), \quad (5.3)$$

where X_μ is a function depending only on x_μ and

$$A_\mu^{(k)} = \sum_{\substack{1 \leq \nu_1 < \dots < \nu_k \leq n \\ (\nu_i \neq \mu)}} x_{\nu_1}^2 x_{\nu_2}^2 \dots x_{\nu_k}^2, \quad A^{(k)} = \sum_{1 \leq \nu_1 < \dots < \nu_k \leq n} x_{\nu_1}^2 x_{\nu_2}^2 \dots x_{\nu_k}^2, \quad S = \frac{c}{A^{(n)}} \quad (5.4)$$

with a constant c . The CKY tensor is written as [2]

$$h = \frac{1}{2} \sum_{k=0}^{n-1} dA^{(k+1)} \wedge d\psi_k \quad (5.5)$$

with the associated vector $\xi = \partial/\partial\psi_0$. The assumptions (a1), (a2) and (a3) are clearly satisfied. The commuting Killing tensors $K^{(j)}$ and Killing vectors $\eta^{(j)}$ are calculated as [2, 3]

$$K^{(j)} = \sum_{\mu=1}^n A_{\mu}^{(j)} (e^{\mu} e^{\mu} + e^{\mu+n} e^{\mu+n}) + \epsilon A^{(j)} e^{2n+1} e^{2n+1}, \quad (5.6)$$

$$\eta^{(j)} = \frac{\partial}{\partial\psi_j}, \quad (5.7)$$

where $\epsilon = 0$ for $D = 2n$ and 1 for $D = 2n + 1$. The 1-forms $\{e^{\mu}, e^{\mu+n}, e^{2n+1}\}$ are orthonormal bases defined by

$$e^{\mu} = \frac{dx_{\mu}}{\sqrt{Q_{\mu}}}, \quad e^{\mu+n} = \sqrt{Q_{\mu}} \left(\sum_{k=0}^{n-1} A_{\mu}^{(k)} d\psi_k \right), \quad e^{2n+1} = \sqrt{S} \left(\sum_{k=0}^n A^{(k)} d\psi_k \right). \quad (5.8)$$

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A Generating function of $K_{ab}^{(j)}$

In this appendix, we rederive the expression of the generating function of $K^{(j)}$ directly from the definition (2.13).

A.1 Auxiliary operators

It is convenient to introduce auxiliary fermionic creation/annihilation operators:

$$\bar{\psi}^a, \quad \psi_a, \quad a = 1, 2, \dots, D \quad (A.1)$$

such that

$$\{\psi_a, \psi_b\} = 0, \quad \{\bar{\psi}^a, \bar{\psi}^b\} = 0, \quad \{\psi_a, \bar{\psi}^b\} = \delta_a^b. \quad (A.2)$$

Also let

$$\bar{\psi}_a := g_{ab} \bar{\psi}^b, \quad \psi^a := g^{ab} \psi_b. \quad (\text{A.3})$$

$$\{\psi_a, \bar{\psi}_b\} = g_{ab}, \quad \{\psi^a, \bar{\psi}^b\} = g^{ab}. \quad (\text{A.4})$$

The Fock vacuum is defined by

$$\psi_a |0\rangle = 0, \quad \langle 0 | \bar{\psi}^a = 0, \quad a = 1, 2, \dots, D, \quad (\text{A.5})$$

with a normalization

$$\langle 0 | 0 \rangle = 1. \quad (\text{A.6})$$

To a 2-form h

$$h = \frac{1}{2} h_{ab} dx^a \wedge dx^b, \quad (\text{A.7})$$

let us associate the following operators:

$$h_{\bar{\psi}} := \frac{1}{2} h_{ab} \bar{\psi}^a \bar{\psi}^b, \quad (\text{A.8})$$

$$h_{\psi} := \frac{1}{2} h^{ab} \psi_a \psi_b. \quad (\text{A.9})$$

Note that

$$(h_{\bar{\psi}})^j = \frac{1}{(2j)!} h_{a_1 \dots a_{2j}}^{(j)} \bar{\psi}^{a_1} \dots \bar{\psi}^{a_{2j}}. \quad (\text{A.10})$$

$$\begin{aligned} h_{a_1 \dots a_{2j}}^{(j)} &= \langle 0 | \psi_{a_{2j}} \dots \psi_{a_1} (h_{\bar{\psi}})^j | 0 \rangle \\ &= (-1)^j \langle 0 | \psi_{a_1} \dots \psi_{a_{2j}} (h_{\bar{\psi}})^j | 0 \rangle. \end{aligned} \quad (\text{A.11})$$

A.2 The generating function of $A^{(j)}$

Let

$$\begin{aligned} A^{(j)} &:= \frac{1}{(2j)!(j!)^2} (h_{c_1 \dots c_{2j}}^{(j)} h^{(j)c_1 \dots c_{2j}}) \\ &= \frac{(2j)!}{(2^j j!)^2} h^{[a_1 b_1} \dots h^{a_j b_j]} h_{[a_1 b_1} \dots h_{a_j b_j]}. \end{aligned} \quad (\text{A.12})$$

$A^{(j)}$ is nontrivial for $j = 0, 1, \dots, [D/2]$.

Note that

$$\begin{aligned} A^{(j)} &= \frac{1}{(2j)!(j!)^2} h_{c_1 \dots c_{2j}}^{(j)} h^{(j)c_1 \dots c_{2j}} \\ &= \frac{1}{(2j)!(j!)^2} h^{(j)c_1 \dots c_{2j}} \times (-1)^j \langle 0 | \psi_{c_1} \dots \psi_{c_{2j}} (h_{\bar{\psi}})^j | 0 \rangle \\ &= (-1)^j \langle 0 | \frac{(h_{\psi})^j}{j!} \frac{(h_{\bar{\psi}})^j}{j!} | 0 \rangle. \end{aligned} \quad (\text{A.13})$$

Then we have

$$\sum_{j=0}^{[D/2]} A^{(j)} \beta^j = \langle 0 | e^{-\sqrt{\beta} h_\psi} e^{\sqrt{\beta} h_{\bar{\psi}}} | 0 \rangle. \quad (\text{A.14})$$

Let us introduce the vielbein

$$g_{ab} = \delta_{ij} e^i_a e^j_b. \quad (\text{A.15})$$

(We assume the Euclidean signature.)

Let E be the matrix with elements

$$E^i_a = e^i_a. \quad (\text{A.16})$$

Then

$$H^a_b = (E^{-1})^a_i \tilde{H}_{ij} E^j_b, \quad \tilde{H}_{ij} = -\tilde{H}_{ji}. \quad (\text{A.17})$$

Also let

$$\theta^i = e^i_a \psi^a, \quad \bar{\theta}^i = e^i_a \bar{\psi}^a, \quad i = 1, 2, \dots, D. \quad (\text{A.18})$$

Then we have $\theta_i = \theta^i$, $\bar{\theta}_i = \bar{\theta}^i$, and

$$\{\theta_i, \theta_j\} = 0, \quad \{\bar{\theta}_i, \bar{\theta}_j\} = 0, \quad \{\theta_i, \bar{\theta}_j\} = \delta_{ij}, \quad (\text{A.19})$$

for $i, j = 1, 2, \dots, D$. It is well known that any real antisymmetric matrix can be block diagonalized by some orthogonal matrix. Therefore, we can choose the vielbein such that \tilde{H} has a block diagonal form and

$$h_\psi = \sum_{\mu=1}^n \lambda_\mu \theta_\mu \theta_{n+\mu}, \quad h_{\bar{\psi}} = \sum_{\mu=1}^n \lambda_\mu \bar{\theta}_\mu \bar{\theta}_{n+\mu}, \quad (\text{A.20})$$

for $n = [D/2]$. Here we assume that $\lambda_\mu \neq 0$. Note that

$$EQE^{-1} = \text{diag}(\lambda_1^2, \lambda_2^2, \dots, \lambda_n^2, \lambda_1^2, \lambda_2^2, \dots). \quad (\text{A.21})$$

For odd D , the last diagonal entry equals to zero.

Then

$$\begin{aligned} \langle 0 | e^{-\sqrt{\beta} h_\psi} e^{\sqrt{\beta} h_{\bar{\psi}}} | 0 \rangle &= \langle 0 | \prod_{\mu=1}^n (1 - \sqrt{\beta} \lambda_\mu \theta_\mu \theta_{n+\mu}) (1 + \sqrt{\beta} \lambda_\mu \bar{\theta}_\mu \bar{\theta}_{n+\mu}) | 0 \rangle \\ &= \prod_{\mu=1}^n (1 + \beta \lambda_\mu^2) \\ &= \det^{1/2}(I + \beta Q). \end{aligned} \quad (\text{A.22})$$

Here I is the $D \times D$ identity matrix.

We have the generating function of $A^{(j)}$:

$$\sum_{j=0}^{[D/2]} A^{(j)} \beta^j = \det^{1/2}(I + \beta Q) = \det(I + \sqrt{\beta} H) = \det(I - \sqrt{\beta} H). \quad (\text{A.23})$$

A.3 Recursion relations for $K^{(j)}$

The Levi-Civita tensor satisfies

$$\varepsilon^{a_1 \dots a_r c_1 \dots c_{D-r}} \varepsilon_{b_1 \dots b_r c_1 \dots c_{D-r}} = r!(D-r)! \delta_{b_1}^{[a_1} \dots \delta_{b_r}^{a_r]}. \quad (\text{A.24})$$

Using (A.24), we can check that $K_{ab}^{(j)}$ has the following form:

$$K_{ab}^{(j)} = A^{(j)} g_{ab} + \frac{1}{(2j-1)!(j!)^2} h_{ac_1 \dots c_{2j-1}}^{(j)} h^{(j)c_1 \dots c_{2j-1}}{}_b. \quad (\text{A.25})$$

Here $A^{(j)}$ is defined by (A.12).

It is possible to show that

$$\frac{1}{(2j-1)!(j!)^2} h_{ac_1 \dots c_{2j-1}}^{(j)} h^{(j)c_1 \dots c_{2j-1}}{}_b = h_a{}^c K_{cd}^{(j-1)} h^d{}_b. \quad (\text{A.26})$$

In the matrix notation, $K^{(j)}$ satisfies the following recursion relation:

$$K^{(j)} = A^{(j)} I + H K^{(j-1)} H. \quad (\text{A.27})$$

Therefore, we can see that $K^{(j)}$ commutes with H . Thus

$$K^{(j)} = A^{(j)} I - Q K^{(j-1)}. \quad (\text{A.28})$$

With the initial condition

$$K^{(0)} = I, \quad K_{ab}^{(0)} = g_{ab}, \quad (\text{A.29})$$

we easily find that

$$K^{(j)} = \sum_{l=0}^j (-1)^l A^{(j-l)} Q^l, \quad (\text{A.30})$$

or

$$K^{(j)a}{}_b = \sum_{l=0}^j (-1)^l A^{(j-l)} (Q^l)^a{}_b. \quad (\text{A.31})$$

We immediately see that

$$K^{(i)} K^{(j)} = K^{(j)} K^{(i)}. \quad (\text{A.32})$$

Using (A.23), we can see that $K^{(k)} = 0$ for $k = [(D+1)/2]$. Indeed, by setting $\beta = -x^{-1}$,

$$\sum_{j=0}^{[D/2]} (-1)^j A^{(j)} x^{-j} = \det^{1/2}(I - x^{-1}Q) = x^{-D/2} \det^{1/2}(xI - Q). \quad (\text{A.33})$$

For $D = 2k$,

$$\sum_{j=0}^k (-1)^{k-j} A^{(j)} x^{k-j} = (-1)^k \det^{1/2}(xI - Q). \quad (\text{A.34})$$

If we set x to be an eigenvalue of Q , the R.H.S. becomes zero. Therefore, we can see that

$$K^{(k)} = \sum_{l=0}^k (-1)^l A^{(k-l)} Q^l = 0, \quad \text{for } D = 2k. \quad (\text{A.35})$$

Similarly, for $D = 2k - 1$,

$$\sum_{j=0}^{k-1} (-1)^{k-j} A^{(j)} x^{k-j} = (-1)^k x^{1/2} \det^{1/2}(xI - Q). \quad (\text{A.36})$$

Thus

$$K^{(k)} = \sum_{l=1}^k (-1)^l A^{(k-l)} Q^l = 0, \quad \text{for } D = 2k - 1. \quad (\text{A.37})$$

Also note that $A^{(j)} = 0$ for $j \geq [D/2] + 1$. Therefore the recursion relations (A.28) becomes trivial for $j \geq k + 1$ and $K_{ab}^{(j)} = 0$ for $j \geq k$. $K^{(j)}$ can be written as (A.30) for all $j \geq 0$ but are nontrivial only for $j = 0, 1, \dots, k - 1$.

Using (A.30) and (A.23), we can see that the generating function of $K^{(j)}$ is

$$K(\beta) := \sum_{j=0}^{k-1} K^{(j)} \beta^j = \det^{1/2}(I + \beta Q) (I + \beta Q)^{-1}. \quad (\text{A.38})$$

A.4 Proof of (A.26)

The L.H.S. of (A.26) is

$$\begin{aligned}
& \frac{1}{(2j-1)!(j!)^2} h_{ac_1 \dots c_{2j-1}}^{(j)} h^{(j)c_1 \dots c_{2j-1}}{}_b \\
&= \frac{1}{(2j-1)!(j!)^2} h^{(j)c_1 \dots c_{2j-1}}{}_b \times (-1)^j \langle 0 | \psi_a \psi_{c_1} \dots \psi_{c_{2j-1}} (h_{\bar{\psi}})^j | 0 \rangle \\
&= \frac{(-1)^{j-1}}{(2j-1)!(j!)^2} h^{(j)c_1 \dots c_{2j-1}}{}_b \langle 0 | \psi_{c_1} \dots \psi_{c_{2j-1}} \psi_a (h_{\bar{\psi}})^j | 0 \rangle \\
&= \frac{(-1)^{j-1}}{(2j)!(j!)^2} h^{(j)c_1 \dots c_{2j}} \langle 0 | \psi_{c_1} \dots \psi_{c_{2j}} \bar{\psi}_b \psi_a (h_{\bar{\psi}})^j | 0 \rangle \\
&= (-1)^{j-1} \langle 0 | \frac{(h_\psi)^j}{j!} \bar{\psi}_b \psi_a \frac{(h_{\bar{\psi}})^j}{j!} | 0 \rangle.
\end{aligned} \tag{A.39}$$

Then

$$\begin{aligned}
K_{ab}^{(j)} &= (-1)^j g_{ab} \langle 0 | \frac{(h_\psi)^j}{j!} \frac{(h_{\bar{\psi}})^j}{j!} | 0 \rangle - (-1)^j \langle 0 | \frac{(h_\psi)^j}{j!} \bar{\psi}_b \psi_a \frac{(h_{\bar{\psi}})^j}{j!} | 0 \rangle \\
&= (-1)^j \langle 0 | \frac{(h_\psi)^j}{j!} [\{\psi_a, \bar{\psi}_b\} - \bar{\psi}_b \psi_a] \frac{(h_{\bar{\psi}})^j}{j!} | 0 \rangle \\
&= (-1)^j \langle 0 | \frac{(h_\psi)^j}{j!} \psi_a \bar{\psi}_b \frac{(h_{\bar{\psi}})^j}{j!} | 0 \rangle.
\end{aligned} \tag{A.40}$$

Thus

$$K_{ab}^{(j)} = (-1)^j \langle 0 | \frac{(h_\psi)^j}{j!} \psi_a \bar{\psi}_b \frac{(h_{\bar{\psi}})^j}{j!} | 0 \rangle. \tag{A.41}$$

Note that

$$[\psi_a, h_{\bar{\psi}}] = h_{aa'} \bar{\psi}^{a'}, \tag{A.42}$$

$$\psi_a (h_{\bar{\psi}})^j | 0 \rangle = j h_a{}^{a'} \bar{\psi}_{a'} (h_{\bar{\psi}})^{j-1} | 0 \rangle, \tag{A.43}$$

$$[h_\psi, \bar{\psi}_b] = \psi_{b'} h^{b'}{}_b, \tag{A.44}$$

$$\langle 0 | (h_\psi)^j \bar{\psi}_b = j \langle 0 | (h_\psi)^{j-1} \psi_{b'} h^{b'}{}_b. \tag{A.45}$$

Then

$$\begin{aligned}
(\text{L.H.S. of (A.26)}) &= \frac{1}{(2j-1)!(j!)^2} h_{ac_1 \dots c_{2j-1}}^{(j)} h^{(j)c_1 \dots c_{2j-1}}{}_b \\
&= (-1)^{j-1} \langle 0 | \frac{(h_\psi)^j}{j!} \bar{\psi}_b \psi_a \frac{(h_{\bar{\psi}})^j}{j!} | 0 \rangle \\
&= h_a{}^{a'} (-1)^{j-1} \langle 0 | \frac{(h_\psi)^{j-1}}{(j-1)!} \psi_{b'} \bar{\psi}_{a'} \frac{(h_{\bar{\psi}})^{j-1}}{(j-1)!} | 0 \rangle h^{b'}{}_b \\
&= h_a{}^{a'} K_{a'b'}^{(j-1)} h^{b'}{}_b \\
&= (\text{R.H.S. of (A.26)}).
\end{aligned} \tag{A.46}$$

This completes the proof of (A.26).

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